ON A CERTAIN CLASS OF QUARTIC CURVES.*

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The object of this paper is the discussion of a class of quartic curves whose equation may be represented thus: Set

$$(1) \qquad m_i=c_i\sqrt{[(x-a_i)^2+(y-b_i)^2]}, \quad i=1, 2, 3,$$

in which $c_i>0$ and the radical is to be taken with the positive sign. Then put

$$(2) \qquad (m_1+m_2+m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)(m_1-m_2-m_3)=0.$$

It is evident that if the factors of the first member of (2) are multiplied together the resulting equation is without radicals and is of the fourth degree in x, y. Therefore its locus is a quartic curve, except in the special case in which the reduced equation breaks up into rational factors.

It should be pointed out that in the equation in its expanded form each of the c's enters to an even power, and consequently the assumption of c_i positive is no limitation on the generality of equation (2). A similar remark applies to the assumption that the radicals are positive.

Attention will be confined to a discussion of the general nature of the locus and the problem of the construction of the curve by continuous motion when the c's are commensurable.

§1. General Nature of the Locus.

Equation (2) is evidently satisfied if any one of the factors of its first member is zero. But, since c_i and the radical in (1) each is to be taken positive, it is clear that

$$m_1+m_2+m_3=0$$

can be satisfied only when $x-a_i=0$ and $y-b_i=0$; that is, when $x=a_1=a_2=a_3$, $y=b_1=b_2=b_3$. The locus of (2) is simply a single point and is therefore not properly a locus of the fourth order. We shall therefore exclude the case from further consideration.

Each of the other three cases may be represented by

$$(3) \qquad m_\lambda=m_\mu+m_\nu, \qquad \lambda,\ \mu,\ \nu=1, 2, 3 \text{ in some order.}$$

It is evident at once from (1) that the branch which is represented by (3) has this property: If P is any point on the branch, then c_λ times its distance

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from (a_λ, b_λ) is equal to c_μ times its distance from (a_μ, b_μ) plus c_v times its distance from (a_v, b_v). Since the c's are to be taken positive and the a's, b's, and c's are to be considered finite it is evident that this condition can always be satisfied whatever the positions of the three points and the values of the c's. Hence the locus of (2) has in general three branches, and these must be distinct except for special values.

Now suppose that any two branches, say,

$$m_\lambda = m_\mu + m_v , \quad m_\mu = m_\lambda + m_v ,$$

have some point in common. Then solving their equations as simultaneous, we have $m_\lambda = m_\mu$; and therefore each of the branches reduces to $m_v = 0$, an equation which can be satisfied only by the single point (a_v, b_v). That is, the two branches of the locus coincide and consist of a single point only. But, since $m_\lambda = m_\mu$, equation (2) now reduces to the form

$$(2m_\lambda + m_v)(2m_\lambda - m_v)(m_v)^2 = 0, \tag{4}$$

the locus of which is evidently not properly of the fourth degree. The case, therefore, in which two branches of the locus of (2) may have a common point is to be excluded from further discussion.

We propose now to find a necessary condition for the existence of an infinite branch. Suppose that

$$m_\lambda = m_\mu + m_v , \qquad \lambda,\ \mu,\ \nu = 1,\ 2,\ 3 \text{ in some order,} \tag{5}$$

is such a branch. Then either x or y is infinite or both are infinite for some point of the locus. Now divide equation (5) by $\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}$; in the result consider the case for which some point P is infinitely removed from the origin. The limit of each of the fractions of the form

$$\frac{\sqrt{[(x-a_\mu)^2+(y-b_\mu)^2]}}{\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}}$$

is easily shown to be 1 when either x or y approaches infinity or when both approach infinity. Passing to the point P and taking the limiting values of the fractions in the equation which results from the suggested division in (5), we have

$$c_\lambda = c_\mu + c_v , \tag{6}$$

a condition which is necessary if the locus of (5) is an infinite branch. Then, evidently, *the locus of* (2) *can in no case have more than one infinite branch,*

and for the existence of such a branch it is necessary and sufficient that a relation (6) *shall hold.*

We consider now the relative position of the closed branches of the curve. Since no two branches can have a point in common, it follows that a closed branch cannot lie partly within and partly without another branch. Suppose that one of the closed branches lies entirely within another. Then draw a line through any point within the inner of these two ovals and through some point on the third branch (considered either as an oval or as an infinite branch). Such a line cuts the quartic curve in five points; and this is impossible. Therefore, *one of the closed branches cannot lie within the other.* A similar discussion leads to the theorem that no parts of two closed branches can lie on the same straight line with any point of the third branch. The facts of this paragraph will enable one to obtain an idea of the form of the locus in each of the two possible cases which may arise.

§2. Construction by Continuous Motion.

In this section the discussion is confined to the case in which the c's are commensurable.

Since the equation of any branch may be written in the form

$$m_\lambda = m_\mu + m_\nu \,, \tag{7}$$

it follows that the problem of construction by continuous motion is completely solved when any branch in its most general case has been constructed.

Since the c's are now to be considered commensurable, we may multiply them by some common number d so that the resulting numbers are integers. Then let $dc_i = k_i$, k_i an integer. Consider the construction of the branch whose equation is of the form

$$dm_1 = dm_2 + dm_3. \tag{8}$$

The coefficient of each radical in the equation is now an integer. The radicals represent the distance of a point P on the locus from A, B, and C, respectively; A, B, C being in order the points (a_1, b_1), (a_2, b_2), (a_3, b_3). Let smooth pegs be placed at A, B, and C.

Now place a pencil at P. Take a cord of convenient length and pass it around the pencil at P and the peg at A and attach it either to the pencil point or to the peg so that $dc_1 = k_1$ plies extend from P to A. (The cord is attached to the pencil if k_1 is even; to the peg, if k_1 is odd.) The unattached end passes out by the pencil at P in a convenient direction.

Take a second cord having the free end in the same direction as the free end of the first cord. Pass the other end around the pencil at P and the peg at B, leaving it yet unattached, so that $dc_2 = k_2$ plies extend from P

to B. If k_2 is even the cord passed last to the pencil; then pass it around the point C making k_3 plies between P and C, and attach the end to the pencil or to the peg at C according as k_3 is even or odd. But if k_2 is odd, the cord passed last to the peg at B; then let it pass from B to C, and then from C around the pencil at P until the requisite number, k_3, of plies extend from C to P; finally attach the end to the pencil or to the peg at C according as k_3 is odd or even.

Now let both cords be stretched tight while the pencil is held firmly at P. Then tie the free ends of the cords together at some convenient distance from the pencil so that when a pull is made on the knot both strings will be drawn tight throughout their entire lengths, with the exception of course of the free ends beyond the knot. Then if the pencil moves and the cords are kept always in the position which has been defined, it is evident that the pencil point describes the branch in consideration; for k_1 times the distance from A remains always equal to k_2 times the distance from B plus k_3 times the distance from C.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

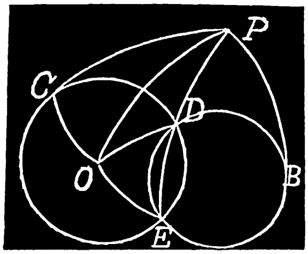
GEOMETRY.

20. Proposed by DR. GEORGE BRUCE HALSTED, Greeley, Colo.

Demonstrate by pure spherical geometry that spherical tangents from any point in the produced spherical chord common to two intersecting circles on a sphere are equal.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

No solution of this problem has yet appeared in the MONTHLY. A simple geometrical solution such as is possible for the corresponding problem in planes is not possible for this problem. The following solution is quite simple.



Let P be point on the common chord DE; PB, PC the tangents, O the pole of one circle. Let $PE=R$, $PD=r$, $PC=\rho$, $PB=\rho'$, $PO=\delta$, $OD=OE=OC=\beta$, $\angle EPO=\phi$.

Then $\cos\beta=\cos R\cos\delta+\sin R\sin\delta\cos\phi\ldots(1)$,

$\cos\beta=\cos r\cos\delta+\sin r\sin\delta\cos\phi\ldots(2)$,

$\cos\delta=\cos\beta\cos\rho\ldots(3)$,

$\cos\phi$ from (1) in (2) gives $\cos\beta(\sin R-\sin r)=\cos\delta\sin(R-r)\ldots(4)$.

$\cos\delta$ from (3) in (4) gives $\cos\rho=(\sin R-\sin r)/\sin(R-r)$.